C4 Integration Exam Question Answers

1

a)

$$\int x e^{x} \, dx$$

u	v
$\frac{du}{dx}$	$\frac{dv}{dx}$

x	e^{x}
1	e^{x}

$$uv - \int v \frac{du}{dx} \, dx$$

$$= x e^{x} - \int e^{x} (1) \, dx$$

$$= x e^{x} - \int e^{x} \, dx$$

$$= x e^{x} - e^{x} + c$$

1

b)

$$\int x^2 e^x \, dx$$

u	v
$\frac{du}{dx}$	$\frac{dv}{dx}$

x^2	e^x
$2x$	e^x

$$uv - \int v \frac{du}{dx} \, dx$$

$$= x^2 e^x - \int e^x (2x) \, du$$

$$= x^2 e^x - 2\int x e^x \, du$$

answer from part a)

$$= x^2 e^x - 2\left[x e^x - e^x\right] + c$$

$$= x^2 e^x - 2xe^x + 2e^x + c$$

(2)

$$\int_0^{\pi/2} e^{(\cos x)+1} \sin x \, dx$$

$u = (\cos x) + 1$ $\qquad \frac{du}{dx} = -\sin x$

hence

$$dx = \frac{1}{-\sin x} du$$

change limits

$x = \frac{\pi}{2}$ $u = 1$

$x = 0$ $u = 2$

$$\int_2^1 e^u \sin x \frac{1}{-\sin x} du$$

$$= \int_2^1 e^u \; -1 \; du$$

$$= -1 \int_2^1 e^u \, du$$

$$= -1 \left[e^u \right]_2^1 = \left[-e^u \right]_2^1 = \left[-e \right] - \left[-e^2 \right]$$

$$= -e + e^2 = e(-1+e) = e(e-1)$$

3

$$\int_0^1 \frac{2^x}{(2^x+1)^2}\,dx$$

$$u = 2^x \qquad \frac{du}{dx} = 2^x \ln 2$$

hence

$$dx = \frac{1}{2^x \ln 2}\,du$$

change limits

$x=1$ $u=2$

$x=0$ $u=1$

$$\int_1^2 2^x \frac{1}{(2^x+1)^2} \frac{1}{2^x \ln}\,du$$

$$= \frac{1}{\ln 2}\int_1^2 2^x \frac{1}{(2^x+1)^2} \frac{1}{2^x}\,du$$

$$= \frac{1}{\ln 2}\int_1^2 u \frac{1}{(u+1)^2} \frac{1}{u}\,du$$

$$= \frac{1}{\ln 2}\int_1^2 \frac{1}{(u+1)^2}\,du$$

③ continued

$$\frac{1}{\ln 2}\int_1^2 (u+1)^{-2}\,du$$

$$= \frac{1}{\ln 2}\left[\frac{1}{1}\ \frac{(u+1)^{-1}}{-1}\right]_1^2$$

$$= \frac{1}{\ln 2}\left[\frac{-1}{(u+1)}\right]_1^2$$

$$= \frac{1}{\ln 2}\left(\left[\frac{-1}{3}\right]-\left[\frac{-1}{2}\right]\right)$$

$$= \frac{1}{\ln 2}\left(\frac{1}{6}\right)$$

$$= \frac{1}{6\ln 2}$$

(4)

a) $$\text{Volume} = \pi \int y^2 \, dx$$

$$\text{Volume} = \pi \int_{-1/4}^{1/2} \left(\frac{1}{3(1+2x)} \right)^2 dx$$

$$= \pi \int_{-1/4}^{1/2} \frac{1}{9(1+2x)^2} \, dx$$

$$= \frac{\pi}{9} \int_{-1/4}^{1/2} (1+2x)^{-2} \, dx$$

$$= \frac{\pi}{9} \left[\frac{1}{2} \frac{(1+2x)^{-1}}{-1} \right]_{-1/4}^{1/2}$$

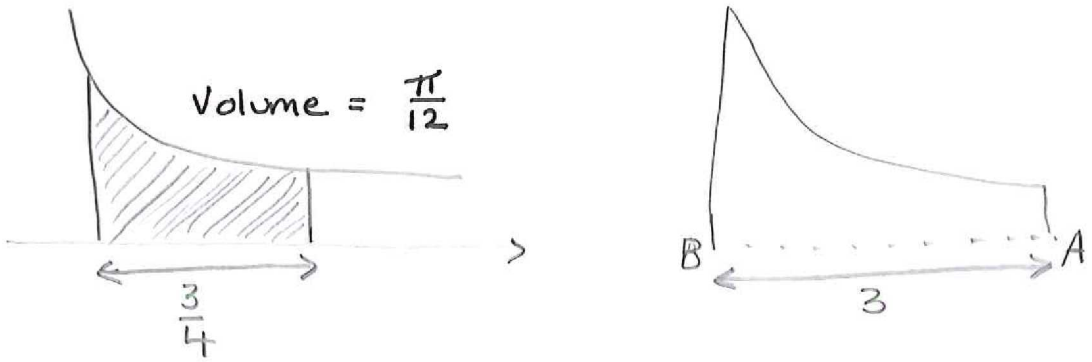
$$= \frac{\pi}{9} \left[\frac{1}{2} \frac{1}{(1+2x)} \right]_{-1/4}^{1/2}$$

$$= \frac{\pi}{9} \left(\left[-\frac{1}{4} \right] - \left[-1 \right] \right)$$

$$= \frac{\pi}{9} \left(\frac{3}{4} \right) = \frac{3\pi}{36} = \frac{\pi}{12}$$

④

b)



SF length $= 3 \div \frac{3}{4} = 4$

SF Volume $= 4^3 = 64$

Volume $= 64 \times \frac{\pi}{12}$

$= \frac{64\pi}{12}$

$= \frac{16\pi}{3}$

5

a)

$$\int \frac{9x+6}{x}\,dx$$

$$= \int \frac{9x}{x} + \frac{6}{x}\,dx = \int 9 + \frac{6}{x}\,dx$$

$$= 9x + 6\ln x + c$$

b)

$$\frac{dy}{dx} = \frac{(9x+6)}{x}\,y^{1/3}$$

$$\int \frac{1}{y^{1/3}}\,dy = \int \frac{9x+6}{x}\,dx$$

$$\frac{y^{2/3}}{2/3} = 9x + 6\ln x + c$$

$$\frac{3}{2}y^{2/3} = 9x + 6\ln x + c$$

$y = 8 \quad x = 1$

$$\frac{3}{2}\sqrt[3]{8}^{\,2} = 9(1) + 6\ln 1 + c$$

$$6 = 9 + 0 + c$$

$$c = -3$$

(5)

b) continued

$$\frac{3}{2}y^{2/3} = 9x + 6\ln x - 3$$

$$3y^{2/3} = 18x + 12\ln x - 6$$

$$y^{2/3} = 6x + 4\ln x - 2$$

$$\left(y^{2/3}\right)^3 = \left(6x + 4\ln x - 2\right)^3$$

$$y^2 = \left(6x + 4\ln x - 2\right)^3$$

(6)

a) $y = 3\cos\left(\frac{\frac{9\pi}{8}}{3}\right) = 1.14805$

b) $\text{Area} = \frac{1}{2}\left(\frac{3\pi}{8}\right)\left[3 + 2(2.7716 + 2.12132 + 1.14805) + 0\right]$

$\text{Area} = \frac{1}{2}\left(\frac{3\pi}{8}\right)\left[15.08202\right]$

c) $\int_0^{\frac{3\pi}{2}} 3\cos\left(\frac{x}{3}\right)\, dx$

$= \left[\frac{3\sin\left(\frac{x}{3}\right)}{1/3}\right]_0^{\frac{3\pi}{2}}$

$= \left[9\sin\left(\frac{x}{3}\right)\right]_0^{\frac{3\pi}{2}}$

$= [9] - [0]$

$= 9$

7

a)

$$\int (5-x)^{1/2}$$

$$= \frac{1}{-1} \frac{(5-x)^{3/2}}{3/2}$$

$$= -\frac{2}{3}(5-x)^{3/2} + C$$

b)

$$\int (x-1)(5-x)^{1/2}$$

u	v		$(x-1)$	$-\frac{2}{3}(5-x)^{3/2}$
$\frac{du}{dx}$	$\frac{dv}{dx}$		1	$(5-x)^{1/2}$

$$uv - \int v \frac{du}{dx} \; dx$$

$$= (x-1)\; -\frac{2}{3}(5-x)^{3/2} - \int -\frac{2}{3}(5-x)^{3/2}(1)\; dx$$

$$= -\frac{2}{3}(x-1)(5-x)^{3/2} - \left[-\frac{2}{3}(-1)\frac{(5-x)^{5/2}}{5/2}\right]$$

$$= -\frac{2}{3}(x-1)(5-x)^{3/2} - \left[\frac{4}{15}(5-x)^{5/2}\right]$$

$$= -\frac{2}{3}(x-1)(5-x)^{3/2} - \frac{4}{15}(5-x)^{5/2} + C$$

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b) ii) $$\left[-\frac{2}{3}(x-1)\sqrt{(5-x)}^{3} - \frac{4}{15}\sqrt{(5-x)}^{5}\right]_{1}^{5}$$

$$= \left[-\frac{2}{3}(4)\sqrt{0}^{3} - \frac{4}{15}\sqrt{0}^{5}\right]$$

$$- \left[-\frac{2}{3}(0)\sqrt{4}^{3} - \frac{4}{15}\sqrt{4}^{5}\right]$$

$$= [0] - \left[0 - \frac{128}{15}\right]$$

$$= \frac{128}{5}$$

(8)

a) $4 - 2x = A(x+1)(x+3) + B(2x+1)(x+3) + C(2x+1)(x+1)$

let $x = -1$

$4 - 2(-1) = A(0)(2) + B(-1)(2) + C(-1)(0)$

$6 = -2B$

$B = -3$

Let $x = -3$

$4 - 2(-3) = A(-2)(0) + B(-5)(0) + C(-5)(-2)$

$10 = 10C$

$C = 1$

let $x = -\frac{1}{2}$

$4 - 2(-\frac{1}{2}) = A(\frac{1}{2})(\frac{5}{2}) + B(0)(\frac{5}{2}) + C(0)(\frac{1}{2})$

$5 = \frac{5}{4}A$

$A = 4$

$A = 4 \quad B = -3 \quad C = 1$

⑧

b)i $$\int \frac{4}{2x+1} + \frac{-3}{x+1} + \frac{1}{x+3}\,dx$$

$$= \int 4\,\frac{1}{2x+1} + -3\,\frac{1}{x+1} + 1\,\frac{1}{x+3}\,dx$$

$$= 4\left(\tfrac{1}{2}\right)\ln(2x+1) - 3\left(\tfrac{1}{1}\right)\ln(x+1) + 1\ln(x+3) + C$$

$$= 2\ln(2x+1) - 3\ln(x+1) + \ln(x+3) + C$$

b)ii
$$\Big[2\ln(2x+1) - 3\ln(x+1) + \ln(x+3)\Big]_0^2$$

$$= \big[2\ln(5) - 3\ln(3) + \ln(5)\big] - \big[2\ln(1) - 3\ln(1) + \ln(3)\big]$$

$$= \big[\ln 25 - \ln 27 + \ln 5\big] - \big[0 - 0 + \ln 3\big]$$

$$= \big[\ln \tfrac{25}{27} + \ln 5\big] - \big[\ln 3\big]$$

$$= \big[\ln \tfrac{125}{27}\big] - \big[\ln 3\big]$$

$$= \ln \tfrac{125}{27} \div 3 = \ln \tfrac{125}{81}$$

9 c)

$$\int \frac{e^{3x}}{1+e^x}\,dx$$

$u = 1 + e^x$ $\quad\frac{du}{dx} = e^x$ $\quad\therefore\ dx = \frac{1}{e^x}\,du$

$$= \int \frac{e^{3x}}{u}\,\frac{1}{e^x}\,du$$

$u = 1 + e^x$

$\therefore\ e^x = u - 1$

$$= \int \frac{(u-1)^3}{u}\,\frac{1}{(u-1)}\,du$$

$$= \int \frac{(u-1)^2}{u}\,du$$

$$= \int \frac{u^2-2u+1}{u}\,du$$

$$= \int \frac{u^2}{u} - \frac{2u}{u} + \frac{1}{u}\,du$$

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c) continued

$$= \int u - 2 + \frac{1}{u} \, du$$

$$= \frac{u^2}{2} - 2u + \ln u + c$$

sub in $u = 1 + e^x$

$$= \frac{(1+e^x)^2}{2} - 2(1+e^x) + \ln(1+e^x) + c$$

$$= \frac{1}{2}(1+e^x)(1+e^x) - 2 \quad 2e^x + \ln(1+e^x) + c$$

$$= \frac{1}{2}(1 + 2e^x + e^{2x}) - 2 - 2e^x + \ln(1+e^x) + c$$

$$= \frac{1}{2} + e^x + \frac{1}{2}e^{2x} - 2 - 2e^x + \ln(1+e^x) + c$$

$$= -e^x + \frac{1}{2}e^{2x} + \ln(1+e^x) - \frac{3}{2} + c$$

$$= \frac{1}{2}e^{2x} - e^x + \ln(1+e^x) + k$$

where $k = -\frac{3}{2} + c$