Dreams Pack

Maths Crib Sheets

Pure Year 2

THE STUFF THAT DREAMS ARE MADE OF

Year 2 Pure

Mensuration

Surface area of sphere = $4\pi r^2$

Area of curved surface of cone = $\pi r \times$ slant height

Arithmetic series

$$S_n = \tfrac{1}{2}n(a+l) = \tfrac{1}{2}n[2a+(n-1)d]$$

Binomial series

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{r}a^{n-r}b^r + \ldots + b^n \quad (n \in \mathbb{N})$$

where $\binom{n}{r} = {}^nC_r = \dfrac{n!}{r!(n-r)!}$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1\times 2}x^2 + \ldots + \frac{n(n-1)\ldots(n-r+1)}{1\times 2\times\ldots\times r}x^r + \ldots \quad (|x|<1, n \in \mathbb{R})$$

Logarithms and exponentials

$$\log_a x = \frac{\log_b x}{\log_b a}$$

$$e^{x\ln a} = a^x$$

Geometric series

$$S_n = \frac{a(1-r^n)}{1-r}$$

$$S_\infty = \frac{a}{1-r} \text{ for } |r|<1$$

Trigonometric identities

$$\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B$$

$$\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A\tan B} \quad (A \pm B \neq (k+\tfrac{1}{2})\pi)$$

$$\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$$

$$\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$$

$$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$$

$$\cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$$

Small angle approximations

$\sin\theta \approx \theta$

$\cos\theta \approx 1 - \dfrac{\theta^2}{2}$

$\tan\theta \approx \theta$

where θ is measured in radians

① $\sec\theta = \dfrac{1}{\cos\theta}$

② $\operatorname{cosec}\theta = \dfrac{1}{\sin\theta}$

③ $\cot\theta = \dfrac{1}{\tan\theta} = \dfrac{\cos\theta}{\sin\theta}$

④ $\sin^2\theta + \cos^2\theta = 1$

⑤ $1 + \tan^2\theta = \sec^2\theta$

⑥ $1 + \cot^2\theta = \operatorname{cosec}^2\theta$

Double Angle

① $\sin 2A = 2\sin A\cos A$

② $\cos 2A = \cos^2 A - \sin^2 A$
$= 1 - 2\sin^2 A$
$= 2\cos^2 A - 1$

③ $\tan 2A = \dfrac{2\tan A}{1-\tan^2 A}$

Express

$a\sin\theta \pm b\cos\theta = R\sin(\theta \pm \alpha)$

$a\cos\theta \pm b\sin\theta = R\cos(\theta \mp \alpha)$

$R = \sqrt{a^2+b^2}$

$\tan\alpha = \dfrac{b}{a}$ or $\dfrac{a}{b}$

Differentiation

First Principles

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$f(x)$	$f'(x)$
$\tan kx$	$k\sec^2 kx$
$\sec kx$	$k\sec kx\tan kx$
$\cot kx$	$-k\operatorname{cosec}^2 kx$
$\operatorname{cosec} kx$	$-k\operatorname{cosec} kx\cot kx$
$\dfrac{f(x)}{g(x)}$	$\dfrac{f'(x)g(x)-f(x)g'(x)}{(g(x))^2}$

$y = e^{f(x)}$ $\quad \dfrac{dy}{dx} = f'(x)e^{f(x)}$

$y = \ln f(x)$ $\quad \dfrac{dy}{dx} = \dfrac{f'(x)}{f(x)}$

$y = a^{f(x)}$ $\quad \dfrac{dy}{dx} = f'(x)a^{f(x)}\ln a$

Year 2 Pure

Integration (+ constant)

$\mathrm{f}(x)$	$\int \mathrm{f}(x)\,\mathrm{d}x$
$\sec^2 kx$	$\frac{1}{k}\tan kx$
$\tan kx$	$\frac{1}{k}\ln\lvert\sec kx\rvert$
$\cot kx$	$\frac{1}{k}\ln\lvert\sin kx\rvert$
$\operatorname{cosec} kx$	$-\frac{1}{k}\ln\lvert\operatorname{cosec} kx + \cot kx\rvert, \quad \frac{1}{k}\ln\lvert\tan(\tfrac{1}{2}kx)\rvert$
$\sec kx$	$\frac{1}{k}\ln\lvert\sec kx + \tan kx\rvert, \quad \frac{1}{k}\ln\lvert\tan(\tfrac{1}{2}kx + \tfrac{1}{4}\pi)\rvert$

$$\int u\frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x = uv - \int v\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x$$

Numerical Methods

The trapezium rule: $\int_a^b y\,\mathrm{d}x \approx \frac{1}{2}h\{(y_0 + y_n) + 2(y_1 + y_2 + \ldots + y_{n-1})\}$, where $h = \frac{b-a}{n}$

The Newton-Raphson iteration for solving $\mathrm{f}(x) = 0 : x_{n+1} = x_n - \frac{\mathrm{f}(x_n)}{\mathrm{f}'(x_n)}$

Year 2 Pure Chapter 1 – Algebraic Methods

Partial Fractions (Different Factors)

$$\frac{4x}{(x+1)(x-3)}$$

$$4x \equiv \frac{A}{(x+1)} + \frac{B}{(x-3)}$$

$$4x \equiv A(x-3) + B(x+1)$$

Let $x = 3$ Solve for B

Let $x = -1$ Solve for A

Partial Fractions (Double Factors)

$$\frac{14x}{(2x+1)(x+1)^2}$$

$$\equiv \frac{A}{(2x+1)} + \frac{B}{(x+1)} + \frac{C}{(x+1)^2}$$

$$14x \equiv A(x+1)^2 + B(2x+1)(x+1) + C(2x+1)$$

Let $x = -1$ Solve for C

Let $x = -\frac{1}{2}$ Solve for A

Let $x = 0$ Solve for B

Watch Out for Improper Fractions

Numerator power ≥ Denominator power

$$\frac{3x^2+5x}{(x-1)(x-2)}$$

$3x^2 \geq x^2$

∴ Improper Fraction

Dealing with Improper Fractions

$$\frac{3x^2-3x-2}{(x-1)(x-2)}$$

Step A: Notice that it is improper.

Step B: Long Division.

$3x^2-3x-2 \div x^2-3x+2$

Whole Number 3

Remainder $6x-8$

Step C: Write as a mixed fraction.

$$3 + \frac{6x-8}{(x-1)(x-2)}$$

Step D: Now split into a partial fraction.

$$3 + \frac{A}{(x-1)} + \frac{B}{(x-2)}$$

Proof by Contradiction

- Make am ASSUMPTION (Opposite to the true statement)
- Show logical steps to reach contradiction to assumption
- Conclude the original statement is true

Tips and Information

Rational numbers can be written as $\frac{a}{b}$

Irrational numbers cannot be written as $\frac{a}{b}$

(where a and b are integers in it simplest form)

Greatest odd integer is n then the next odd number is $n+2$

If n = even
then any even number is
$n = 2k$

If n = odd
then any odd number is
$n = 2k+1$

Example

Statement: $\sqrt{2}$ is irrational

Assumption: $\sqrt{2}$ is rational

$\sqrt{2} = \frac{a}{b} simplifed$

$2 = \frac{a^2}{b^2}$

$2b^2 = a^2$

$\therefore a^2 = even$

$\therefore a = even$

$2b^2 = a^2$

$2b^2 = (2n)^2$

$2b^2 = 4n^2$

$b^2 = 2n^2$

$\therefore b = even$

a and b are even

∴ contradicts that

$\frac{a}{b}$ *is simplifed*

$\therefore \sqrt{2}$ is irrational

Year 2 Pure Chapter 2 - Functions

Notation

Modulus $|-5| = |5|$

$f(x) = 2|x| + 3$ or

$f : x \mapsto 2|x| + 3$

Mapping

One-to-One = Function

Many-to-One = Function

One-to-Many = **Not** Function

Inverse Functions

Equation $f^{-1}(x)$
Take $f(x)$ and make x the subject then switch the letters x and y

Graphing $f^{-1}(x)$
Reflect $f(x)$ in the line $y = x$

Composite

$gf(x)$
sub x into $f(x)$ then take that answer and sub into g

$fg(x)$
sub x into $g(x)$ then take that answer and sub into f

Graphing Modulus

$y = |f(x)|$
reflect everything below the x axis to above the x axis.

$y = f(|x|)$
reflect everything right of y axis to left of the x axis.

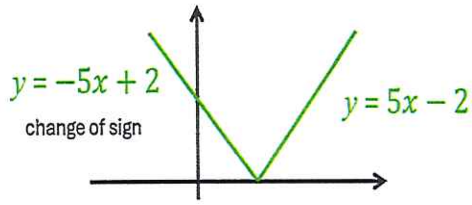
Domain and Range

Range = possible y values
Domain = possible x values

$f(x)$ range = $f^{-1}(x)$ domain
$f(x)$ domain = $f^{-1}(x)$ range

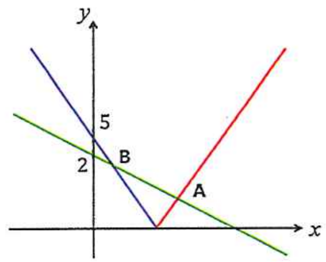
Graphing

$y = |5x - 2|$



Solving Equations

Solve $|3x - 5| = 2 - \frac{1}{2}x$



Solve for solution **A**

$3x - 5 = 2 - \frac{1}{2}x$

$\frac{7}{2}x = 7$

$x = 2$

Solve for solution **B**

$-3x + 5 = 2 - \frac{1}{2}x$

$-\frac{5}{2}x = -3$

$x = \frac{6}{5}$

Transforming Graphs

$f(x + a)$ *left*
$f(x - a)$ *right*

$f(x) + a$ *up*
$f(x + a)$ *down*

$af(x)$ (x, ay)
$f(ax)$ $(\frac{x}{a}, y)$

$-f(x)$ reflect in x axis
$f(-x)$ reflect in x axis

Multiple Transformations

$y = -3|x + 2| + 4$

Plot $|x + 2|$

then transform $(x, -3y)$

then transform $(x, y + 4)$

Year 2 Pure Chapter 3 - Sequences and Series

Arithmetic Sequences

a = 1st term

d = common difference

U_n = nth term

U_1	U_2	U_3	U_n
a	a + d	a + 2d	a + (n − 1)d

Watch Out: Terms and Years

2000	2001	2002	2003
n = 1	n = 2	n = 3	n = 4

Sum of Arithmetic Series

Sequence 5, 7, 9, 11

Series 5 + 7 + 9 + 11

$$S_n = \frac{n}{2}[2a + (n - 1)d]$$

or

$$S_n = \frac{n}{2}[a + l]$$

l = last term

Common Ratio Geometric

U_1	U_2	U_3
2	x	18

$$\frac{x}{2} \qquad \frac{18}{x}$$

common ratio $\frac{x}{2} = \frac{18}{x}$

$$x = 6$$

Geometric Sequences

a = 1st term

r = common ratio

U_n = nth term

U_1	U_2	U_3	U_n
a	ar	ar^2	$ar^{(n-1)}$

Sum of Geometric Series

$$S_n = \frac{a(1 - r^n)}{1 - r}$$

or

$$S_n = \frac{a(r^n - 1)}{1 - r}$$

Sum To Infinity (Geometric)

$$S_n = \frac{a}{1 - r}$$

Sigma Notation

$\sum$ Sum of

$\sum_{1}^{20}$ Sum of 1st 20 terms

Expression to calculate each term.

$\sum 3r + 1$

Prove Arithmetic Sum

$S_n = a + (a + d) + (a + 2d) + \cdots + (a + (n - 1)d)$

Reverse

$S_n = (a + (n - 1)d) + \cdots + (a + 2d) + (a + d) + a$

Adding:

$2S_n = (2a + (n - 1)d) + \cdots + (2a + (n - 1)d)$

$$2S_n = n(2a + (n - 1)d)$$

$$S_n = \frac{n}{2}(2a + (n - 1)d)$$

Prove Geometric Sum

$S_n = a + ar + ar^2 + \cdots + ar^{n-2} + ar^{n-1}$

Multiplying by r

$rS_n = ar + ar^2 + \cdots + ar^{n-2} + ar^{n-1} + ar^n$

Subtracting

$$S_n - rS_n = a - ar^n$$

$$S_n(1 - r) = a(1 - r^n)$$

$$S_n = \frac{a(1 - r^n)}{1 - r}$$

Recurrence Relations

The rules require the previous term to calculate the next.

$$U_{n+1} = U_n + 4$$
$$U_1 = 7$$
$$U_2 = 7 + 4 = 11$$
$$U_2 = 11 + 4 = 15$$

Period Sequences

Repeating in a cycle

2, 3, 4, 2, 3, 5, 2, 3, 5.....

Periodic Order = 3

Repeats every 3 terms

Year 2 Pure Chapter 4 – Binomial Expansion

Expansion A

Expansion for $(a+b)^n$

use when n is positive

Examples:

$(2+x)^5$

$(1+3x)^4$

$(4-2x)^6$

Expansion B

Expansion for $(1+x)^n$

use when n is positive, negative or fractional

Examples

$(1+3x)^{-2}$

$(1+2x)^{\frac{1}{2}}$

$(1+4x)^{-\frac{1}{2}}$

Watch out for when you don't have a 1 in the bracket

$(2+4x)^{-3}$

becomes

$2^{-3}(1+2x)^{-3}$

becomes

$\frac{1}{8}(1+2x)^{-3}$

Now expand and multiple all terms by $\frac{1}{8}$

Identify n and x

$\sqrt{1-3x} = (1-3x)^{\frac{1}{2}}$

$n = \frac{1}{2} \quad x = -3x$

$\frac{1}{(1+2x)^3} = (1+2x)^{-3}$

$n = -3 \quad x = 2x$

$(2+4x)^{-2} = 2^{-3}(1+2x)^{-2}$

$= \frac{1}{4}(1+2x)^{-2}$

$n = -2 \quad x = 2x$

$(1+x)^n$ Terms

1st Term = 1

2nd Term = nx

3rd Term = $\frac{n(n-1)}{2 \times 1}x^2$

4th Term = $\frac{n(n-1)(n-2)}{3 \times 2 \times 1}x^3$

Combining Expansions

$\frac{(1+x)^{\frac{1}{2}}}{(1-x)^{-\frac{1}{2}}} = (1+x)^{\frac{1}{2}}(1-x)^{-\frac{1}{2}}$

Expand individually.

$= (1+\frac{1}{2}x - \frac{1}{8}x^2)(1+\frac{1}{2}x+\frac{3}{8}x^2)$

Multiply out the brackets.

$= 1 + x + \frac{1}{2}x^2 + \cdots$

Partial Fractions and Expansions

$\frac{4-5x}{(1+x)(2-x)}$

$= \frac{A}{(1+x)} + \frac{B}{(2-x)}$

Find A and B

(In the normal way)

$= \frac{3}{(1+x)} - \frac{2}{(2-x)}$

Then rewrite

$= 3(1+x)^{-1} - 2(2-x)^{-1}$

Now expand

$= 3 - 3x + 3x^2 - 1 - \frac{x}{2} - \frac{x^2}{4}$

Simplify

$= 2 - \frac{7x}{2} - \frac{11x^2}{4}$

Expansion Valid or Not

$(1+x)^n$

If n is negative or a fraction

The expansion is only VALID if

$(1+x) \neq 0$

$\therefore [x] < 1$

$(1+x)^{\frac{1}{2}}$ valid if

$[x] < 1$

$(1-2x)^{-2}$ valid if

$[-2x] < 1$

$[x] < -\frac{1}{2}$

$(1-\frac{x}{4})^{-\frac{1}{3}}$ valid if

$\left[\frac{x}{4}\right] < 1$

$[x] < -4$

Year 2 Pure Chapter 5 - Radians

Calculator Mode

Make sure your calculator is in radian mode.

Measurements

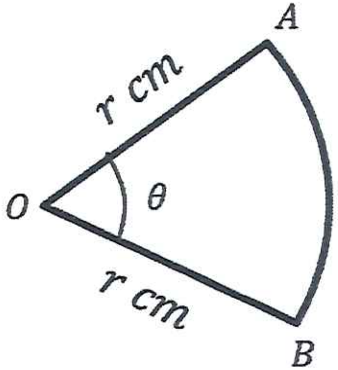
$2\pi = 360°$

$\pi = 180°$

$\frac{\pi}{2} = 90°$

Arcs

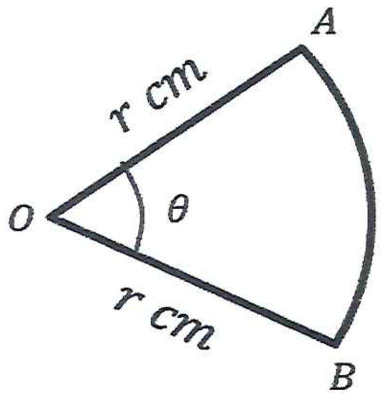
IF θ is in radians then



$$AB = r\theta$$

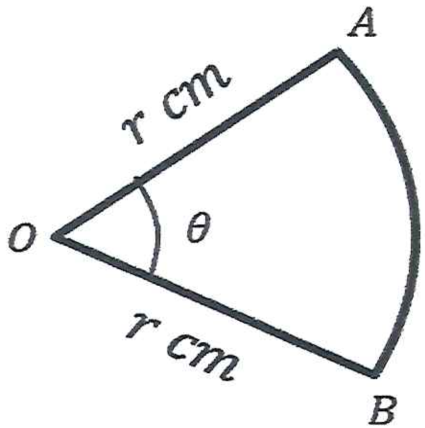
Area of a Sector

IF θ is in radians then



$$\text{Area} = \frac{1}{2}r^2\theta$$

Area of a Triangle



$$\text{Area} = \frac{1}{2}r^2\sin\theta$$

Small Angle Approximations

(Given in the formula booklet)

$$\sin\theta \approx \theta$$

$$\tan\theta \approx \theta$$

$$\cos\theta \approx 1 - \frac{\theta^2}{2}$$

Trig Equations + Radians

Solving $\sin\theta$

1st solution: sin^{-1}

2nd solution: $\pi - 1st\ solution$

All other solution: $\pm 2\pi$

Solving $\cos\theta$

1st solution: cos^{-1}

2nd solution: $2\pi - 1st\ solution$

All other solution: $\pm 2\pi$

Solving $\tan\theta$

1st solution: tan^{-1}

All other solution: $\pm \pi$

Year 2 Pure Chapter 6 + 7 - Trigonometry Functions

Reciprocal Identities

$$\sec x = \frac{1}{\cos x}$$

$$\operatorname{cosec} x = \frac{1}{\sin x}$$

$$\cot x = \frac{1}{\tan x} = \frac{\cos x}{\sin x}$$

3rd letter is a clue to the identity

co**t** x **t** for tan

More Identities

$$1 + tan^2 x = sec^2 x$$

$$1 + cot^2 x = cosec^2 x$$

Quadratic Questions

$$4\, cosec^2 x - cot\, x - 9 = 0$$

$$4(1 + cot^2 x) - cot\, x - 9 = 0$$

$$4 + 4cot^2 x - cot\, x - 9 = 0$$

$$(4\, cot\, x - 5)(cot\, x + 1) = 0$$

Now solve

Double Angle

(Not in the formula booklet)

$$\sin 2A = 2 \sin A \cos A$$

$$\cos 2A = cos^2 A - sin^2 A$$

$$= 2cos^2 A - 1$$

$$= 1 - sin^2 A$$

$$\tan 2A = \frac{2\, tan\, A}{1 - tan^2 A}$$

Addition Formula

(In the formula Booklet)

$\sin(A + B) = \sin A \cos B + \cos A \sin B$
$\sin(A - B) = \sin A \cos B - \cos A \sin B$

$\cos(A + B) = \cos A \cos B - \sin A \sin B$
$\cos(A - B) = \cos A \cos B + \sin A \sin B$

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

Tips: Above formulas can be useful when your see

$\cos(3x) = \cos(2x + x)$
$\sin(3x) = \sin(2x + x)$
$\sin(15) = \sin(45 - 30)$

R Formula

$$R \sin(x \pm \alpha)$$
$$or\ R \cos(x \pm \alpha)$$

work out R $R = \sqrt{a^2 + b^2}$

work out α $\tan \alpha = \frac{b}{a} or \frac{a}{b}$

How to identify a and b

$$\mathbf{3}\, sin\, x + \mathbf{4} \cos x$$

Compare with addition formula:

$$\mathbf{\cos \alpha}\, sin\, x + \mathbf{\sin \alpha} \cos x$$

Hence

$$\cos \alpha = 3\ and\ \sin \alpha = 4$$

$$\therefore \tan \alpha = \frac{4}{3}$$

R Formula (Max and Min)

$\sin x$ maximum value is 1
$\sin x$ minimum value is -1

	max	min
$5 \sin x$	5	−5
$2 + 5 \sin x$	7	−7
$2 + 5 \sin 3x$	7	−7
$-sin\, x$	1	−1
$\sqrt{2}\, sin\, 3x$	$\sqrt{2}$	$-\sqrt{2}$
$1 + \sqrt{2}\, sin\, 3x$	$1 + \sqrt{2}$	$1 - \sqrt{2}$
$2(1 + \sqrt{2}\, sin\, 3x)$	$2 + 2\sqrt{2}$	$2 - 2\sqrt{2}$

$\cos x$ maximum value is 1
$\cos x$ minimum value is -1

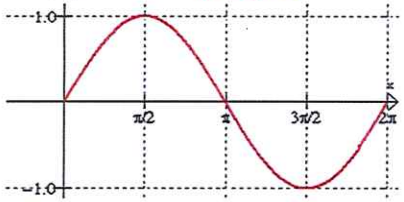
$5 \cos x$	5	−5
$2 + 5 \cos x$	7	−7
$2 + 5 \cos 3x$	7	−7
$-cos\, x$	1	−1
$\sqrt{2}\, cos\, 3x$	$\sqrt{2}$	$-\sqrt{2}$
$1 + \sqrt{2}\, cos\, 3x$	$1 + \sqrt{2}$	$1 - \sqrt{2}$
$2(1 + \sqrt{2}\, cos\, 3x)$	$2 + 2\sqrt{2}$	$2 - 2\sqrt{2}$

Year 2 Pure Chapter 6 + 7 - Trigonometry Functions

R Formula (When it Occurs)

$y = \sin x$

the first max where $x > 0$ happens at (90, 1)



$y = \sin(x - 30)$

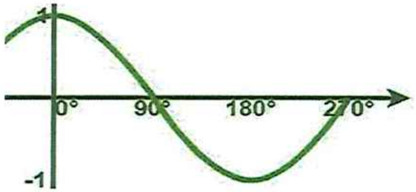
max happens at $(90 + 30, 1)$

$y = \sin(2x - 30)$

max happens at $(\frac{90 + 30}{2}, 1)$

$y = \cos x$

the first max where $x \geq 0$ happens at (0, 1)



$y = \cos(x + 30)$

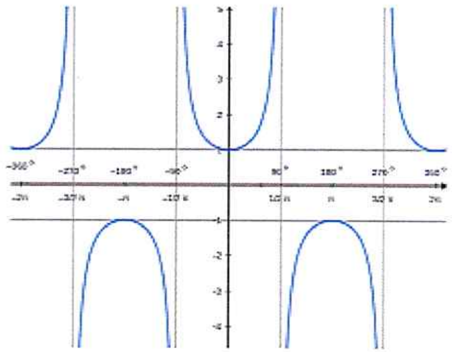
max happens at $(0 - 30, 1)$

$y = \cos(2x - 30)$

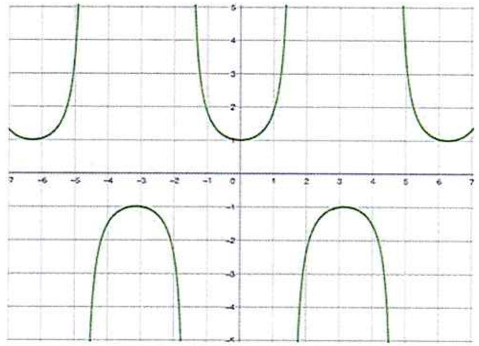
max happens at $(\frac{0 - 30}{2}, 1)$

SEC, COSEC, COT Graphs

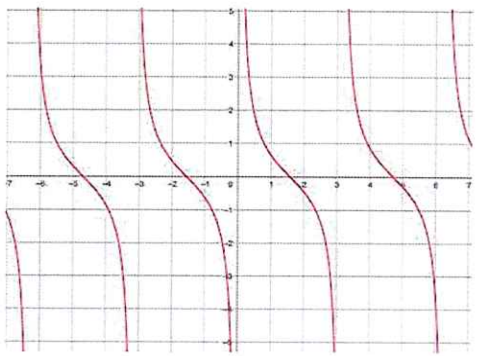
$y = \sec x$



$y = \operatorname{cosec} x$

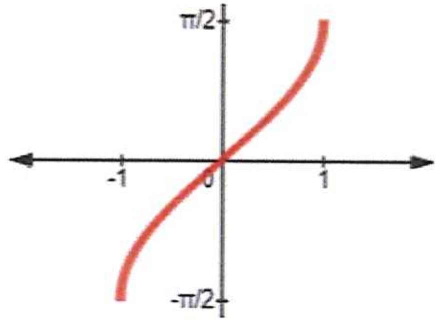


$y = \cot x$

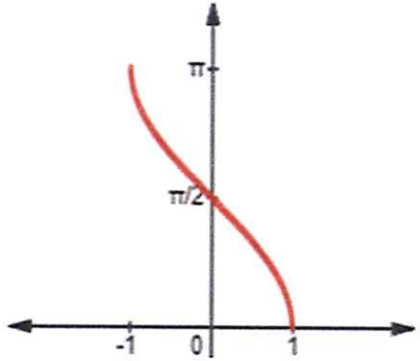


arcsin, arccos, arctan Graphs

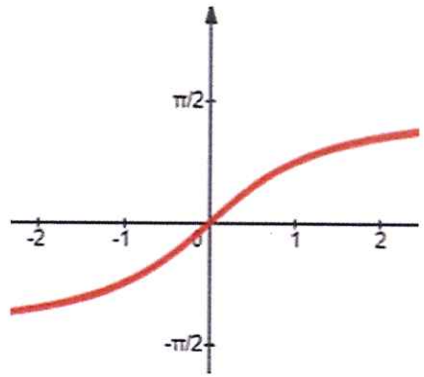
$y = \arcsin x$



$y = \arccos x$



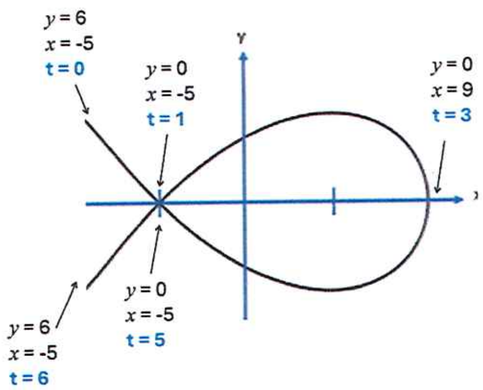
$y = \arctan x$



Year 2 Pure Chapter 8 - Parametric Equations

Parametric Graphs

The x co-ordinate and the y co-ordinate are calculated independently of each other.



x and y co-ordinates are calculated using the t variable

Parametric to Cartesian *without* Trig functions

x equation:
make t the subject.

y equation:
substitute the t equation into the y equation.

Example A

$$x = 2t$$
$$y = t^2$$

Make t the subject:

$$t = \frac{x}{2}$$

Substitute into the y equation:

$$y = \frac{x^2}{4}$$

Parametric to Cartesian *with* Trig functions

Identify a trig identify that connects the x and y equations.

Example A

$$x - 2 = \sin t$$
$$y + 3 = \cos t$$

can be connected by

$$\sin^2 t + \cos^2 t = 1$$

to give

$$(x-2)^2 + (y+3)^2 = 1$$

Example B

$$x = \sin t$$
$$y = \sin 2t$$

can be connected by

$$y = 2 \sin t \cos t$$

and

$$\sin^2 t + \cos^2 t = 1$$

rearrange to

$$\sin t = \sqrt{1 - \cos^2 t}$$

then substitute to give

$$y = 2x\sqrt{1 - \cos^2 t}$$

Year 2 Pure Chapter 9 - Differentiation

Trigonometric Functions

$y = \sin f(x)$

$\frac{dy}{dx} = f'(x) \cos f(x)$

$y = \cos f(x)$

$\frac{dy}{dx} = -f'(x) \sin f(x)$

More Trig Functions

(in the formula booklet)

$\tan kx \rightarrow k \sec^2 kx$

$\sec kx \rightarrow k \sec kx \tan kx$

$\cot kx \rightarrow -k \operatorname{cosec}^2 kx$

$\operatorname{cosec} kx \rightarrow -k \operatorname{cosec} kx \cot kx$

Exponents

$y = e^{f(x)}$

$\frac{dy}{dx} = f'(x)\, e^{f(x)}$

$y = a^{f(x)}$

$\frac{dy}{dx} = f'(x)\, a^{f(x)} \ln a$

Logs

$y = \ln f(x)$

$\frac{dy}{dx} = \frac{f'(x)}{f(x)}$

Chain Rule

(Function within a function)

Examples of chain rule functions

$y = (1 + 2x)^4$

$y = (7 - x)^{\frac{1}{2}}$

$y = e^{\cos x}$

$y = \cos(2x - 1)$

Example

$y = (5x^2 + 2)^4$

$5x^2 + 2$	u^4
$10x$	$4u^3$

$\frac{dy}{dy} = 40xu^3$

$\frac{dy}{dy} = 40x(5x^2 + 2)^3$

Product Rule

(Function multiply a function)

$y = x(1 + 3x)^5$

$y = \sin 2x \cos 3x$

$y = e^x \sin x$

Example

$y = x^2 \sin x$

x^2	$\sin x$
$2x$	$\cos x$

Multiply opposite corners and add.

$\frac{dy}{dy} = x^2 \cos x + 2x \sin x$

Quotient Rule

(Function divided a function)

$y = \frac{5x}{x + 1}$

$y = \frac{\ln x}{x + 1}$

$y = \frac{\sin x}{\ln x}$

Example

$y = \frac{2x}{x^2 - 3}$

$2x$	$x^2 - 3$
2	$2x$

$\frac{f(x)}{g(x)} = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$

given in the formula booklet.

$\frac{dy}{dx} = \frac{2(x^2 - 3) - 2x(2x)}{[x^2 - 3]^2}$

Year 2 Pure Chapter 9 - Differentiation

Parametric Equations

Take the x equations and get

$$\frac{dx}{dt}$$

Then take the reciprocal to get

$$\frac{dt}{dx}$$

Take the y equations and get

$$\frac{dy}{dt}$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

Implicit Differentiation

How to deal y terms:
Differentiate with respect to y and multiply by $\frac{dy}{dx}$

$$y^2 \rightarrow 2y\frac{dy}{dx}$$

Example

$$y^2x$$

y^2	x
$2y\frac{dy}{dx}$	1

$$\frac{dy}{dx} = 1y^2 + 2yx\frac{dy}{dx}$$

Now make $\frac{dy}{dx}$ the subject of the equation

Second Derivatives

Concave
when f(x) is concave
$f''(x) \leq 0$

Convex
when f(x) is concave
$f''(x) \geq 0$

Point of inflection
when f(x) is a point of inflection
$f''(x) = 0$

Rates of Change

Use chain rule to connect rates of change.

Area changing with time

$$= \frac{dA}{dt}$$

$$\frac{dA}{dt} = \frac{dA}{dx} \times \frac{dx}{dt}$$

Volume changing with time

$$= \frac{dA}{dt}$$

$$\frac{dV}{dt} = \frac{dV}{dx} \times \frac{dx}{dt}$$

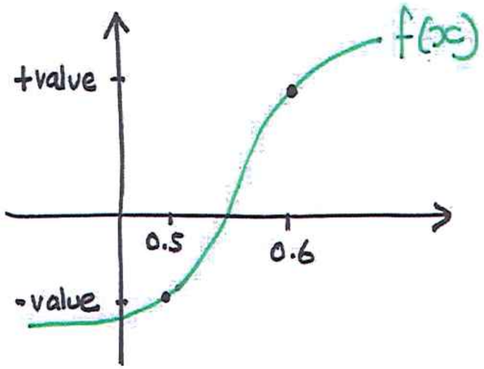
Watch out for decreasing rates of change.

$$\frac{dV}{dt} = -\frac{2v}{15}$$

Negative because decreasing

Year 2 Pure Chapter 10 - Numerical Methods

Locating Roots Between Two Values



f(0.5) = negative value
f(0.6) = positive value

change of sign
and continuous function

∴ root between
0.5 and 0.6

Show a Value is a Possible Root

Root λ = 2.307

Check either side of the number.

Use upper and lower bound values.

f(2.3065) = negative value

f(2.3075) = positive value

change of sign
and continuous function
between
[2.3065 and 2.3075]

so λ = 2.307
is a root to 3d.p.

Iteration

Use the previous answer to locate a more accurate answer.

$x_{n+1} = 2x_n + 4$
$x_1 = 3$
$x_2 = 2(3) + 4 = 10$
$x_3 = 2(10) + 4 = 24$

Newton Raphson

A formula for finding roots.

$$x_{n+1} = x_n - \frac{f(x)}{f'(x)}$$

Given in the formula booklet.

Year 2 Pure Chapter 11 – Integration

Standard Functions (Must Learn)

$\int ax^n\,dx = \frac{ax^{n+1}}{n+1} + c$

$\int (ax+b)^n\,dx = \frac{1}{a(n+1)}(ax+b)^{n+1}$

$\int \frac{1}{ax}\,dx = \frac{1}{a}\ln|ax| + c$

denominator to power of 1

$\int \frac{1}{(ax+b)}\,dx = \frac{1}{a}\ln|ax+b| + c$

denominator to power of that is not 1

$\int \frac{1}{(ax+b)^n}\,dx = \frac{1}{a(-n+1)}(ax+b)^{-n+1}$

$\int e^{ax+b}\,dx = \frac{1}{a}e^{ax+b} + c$

$\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b) + c$

$\int \sin(ax+b)\,dx = -\frac{1}{a}\cos(ax+b) + c$

Trig Function (Formula Booklet)

Integration

$\int sec^2\,kx\,dx = \frac{1}{k}tankx + c$

$\int tan\,kx\,dx = \frac{1}{k}\ln[\sec kx] + c$

$\int cot\,kx\,dx = \frac{1}{k}\ln|\sin kx| + c$

$\int cosec\,kx\,dx$

$= -\frac{1}{k}\ln|cosec\,kx + \cot kx| + c$

$= \frac{1}{k}\ln|tan\left(\frac{1}{2}kx\right)| + c$

$\int sec\,kx\,dx$

$= \frac{1}{k}\ln|sec\,kx + \tan kx| + c$

$= \frac{1}{k}\ln|tan\,(\frac{1}{2}kx + \frac{1}{4}\pi)| + c$

Differentiation

First Principles

$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h}$

f(x) ← Integration	f′(x)
$\frac{1}{k}\tan kx + C$	$k\sec^2 kx$
$\frac{1}{k}\sec kx + C$	$k\sec kx\tan kx$
$\frac{1}{k}\cot kx + C$	$-k\,\text{cosec}^2\,kx$
$\frac{1}{k}\text{cosec}\,kx + C$	$-k\,\text{cosec}\,kx\cot kx$

Reverse Chain Rule (Multiplying Functions)

$\int f'(x)(f(x))^n$

Possible answer

$[f(x)]^{n+1}$

Differentiate possible answer to see if you need a multiplier

Reverse Chain Rule (Dividing Functions)

$\int \frac{f'(x)}{f(x)}\,dx$

Possible answer

$\ln|f(x)| + c$

Differentiate possible answer to see if you need a multiplier

By Parts (Multiplying Functions)

$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$

Given in the formula booklet.

u will always be the x^n term **UNLESS** one term is $\ln x$.

Special Case:

$\int \ln x\,dx = \int(1)(\ln x)\,dx$

Now do by parts

By Substitution

Rewrite the question and the limits in terms of u

$\int_1^2 x\sqrt{2x+5}\,dx$

Let $u = 2x+5$

Rearrange $x = \frac{u-5}{2}$

Differentiate $\frac{du}{dx} = 2$

Rearrange $dx = \frac{1}{2}du$

Change Limits using, $u = 2x+5$

When $x = 2$ then $u = 9$

When $x = 1$ then $u = 7$

Now substitute,

$\int_7^9 (\frac{u-5}{2})\sqrt{u}\,\frac{1}{2}du$

Tidy up,

$\int_7^9 \frac{u^{\frac{3}{2}}}{4} - \frac{5u^{\frac{1}{2}}}{4}\,du$

Now integrate, and substitute in limits.

Year 2 Pure Chapter 11 - Integration

Partial Fractions

Take a single fraction and split it into two or more fractions then integrate each individual fraction.

$$\int \frac{A}{3x+1} + \frac{B}{x-2} + \frac{C}{(x-2)^2}\,dx$$

Numerator: Power of 1

$$\int \frac{A}{3x+1} = \frac{1}{3}A\ln[x-2]$$

$$\int \frac{B}{x-2} = B\ln[x-2]$$

Numerator: Power not 1

$$\int \frac{C}{(x-2)^2} = \frac{C(x-2)^{-1}}{-1}$$

$$= \frac{1}{3}A\ln[x-2] + \frac{1}{3}A\ln[x-2] + \frac{C(x-2)^{-1}}{-1} + c$$

Before doing partial fractions always check to see if you have a **top-heavy fraction**. Is the highest power of the numerator equal or greater than the highest power of the denominator?

$$\int \frac{9x^2-3x+2}{9x^2-4}$$

If yes then do algebraic long division,

$$= \int 1 + \frac{6-3x}{9x^2-4}$$

Now partial fractions,

$$= \int 1 + \frac{1}{3x-2} - \frac{2}{3x-2}$$

Now integrate.

Differential Equations

(Mix of variable and derivatives)

$$\frac{dy}{dx} = xy$$

Split

$$\int \frac{1}{y}\,dy = \int x\,dx$$

Integrate

$$ln\ [y] = \frac{x^2}{2} + c$$

Make y the subject

$$y = e^{\frac{x^2}{2}+c}$$

Simplify if possible

$$y = e^c e^{\frac{x^2}{2}+c}$$

$$y = Ae^{\frac{x^2}{2}+c}$$

Parametric Equations Area Under the Graph

Cartesian Equation

$$Area = \int y\,dx$$

Parametric Equation

$$Area = \int y\,\frac{dx}{dt}\,dt$$

Don't forget to change the limit

$x\ limits\ to\ t\ limits$

Trapezium Rule

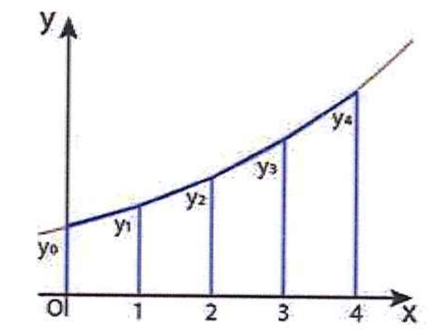
(Estimate Area Under the Graph)

$$\frac{1}{2}h[(y_o + y_1) + 2(y_1 + y_1 + \cdots \ldots + y_{n-1})]$$

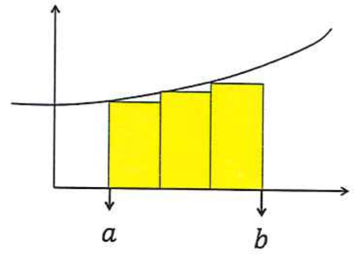
h = width of the trapeziums

$y_o + y_1$ = First and last height

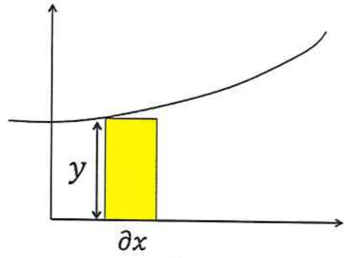
$y_1 + y_1 + \cdots \ldots + y_{n-1}$ = All other heights



Integration as a Limit of a sum



$$\lim_{\partial x \to 0} \sum_{x=a}^{b} Area\ of\ Rectangle$$



$$\lim_{\partial x \to 0} \sum_{x=a}^{b} y\,\partial x$$

This is an integration in disguise.

$$\lim_{\partial x \to 0} \sum_a^b y\,\partial x = \int_a^b y\,dx$$

Year 2 Pure Chapter 11 - Integration

Which Integration Method?

Non-Trig Functions

- Standard Function
- Reverse Chain Rule
- By Parts or By Substitution
- Partial Fractions / Improper Algebraic Fractions

$f(x)$	How to deal with it	$\int f(x)dx$
$\frac{1}{x}$	Standard function	$\ln x$
$\ln x$	Use IBP, where $u = \ln x, \frac{dv}{dx} = \ln x$	$x\ln x - x$
$\frac{x}{x+1}$	Use algebraic division. $\frac{x}{x+1} \equiv 1 - \frac{1}{x+1}$	$x - \ln\lvert x+1\rvert$
$\frac{1}{x(x+1)}$	Use partial fractions.	$\ln\lvert x\rvert - \ln\lvert x+1\rvert$
$\frac{4x}{x^2+1}$	Reverse chain rule.	$2\ln\lvert x^2+1\rvert$
$\frac{x}{(x^2+1)^2}$	Power around denominator so rewrite as $x(x^2+1)^{-2}$ then reverse chain rule	$-\frac{1}{2}(x^2+1)^{-1}$
e^{2x+1}	Divide by coefficient of x	$\frac{1}{2}e^{2x+1}$
$x\sqrt{2x+1}$	IBP or use sensible substitution. $u = 2x+1$ or even better, $u^2 = 2x+1$.	$\frac{1}{15}(2x+1)^{\frac{3}{2}}(3x-1)$

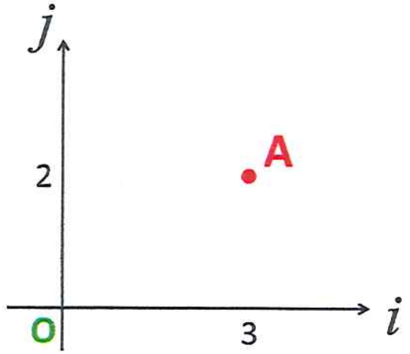
Trig Functions

- Standard Function
- Formula Sheet
- Reverse Chain Rule
- By Parts or By Substitution
- Rewrite using Trig Identities

$f(x)$	How to deal with it	$\int f(x)dx$
$\sin x$	Standard function	$-\cos x$
$\cos x$	Standard function	$\sin x$
$\tan x$	Formula booklet	$\ln\lvert\sec x\rvert$
$\sin^2 x$	Double Angle for $\cos 2x$ $\cos 2x = 1 - 2\sin^2 x$ $\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x$	$\frac{1}{2}x - \frac{1}{4}\sin 2x$
$\cos^2 x$	Double Angle for $\cos 2x$ $\cos 2x = 2\cos^2 x - 1$ $\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x$	$\frac{1}{2}x + \frac{1}{4}\sin 2x$
$\tan^2 x$	$1 + \tan^2 x \equiv \sec^2 x$ $\tan^2 x \equiv \sec^2 x - 1$	$\tan x - x$
$\operatorname{cosec} x$	Formula booklet	$-\ln\lvert\operatorname{cosec} x + \cot x\rvert$
$\sec x$	Formula booklet	$\ln\lvert\sec x + \tan x\rvert$
$\cot x$	Formula booklet	$\ln\lvert\sin x\rvert$
$\operatorname{cosec}^2 x$	By observation.	$-\cot x$
$\sec^2 x$	Formula booklet	$\tan x$
$\cot^2 x$	$1 + \cot^2 x \equiv \operatorname{cosec}^2 x$	$-\cot x - x$
$\sin 2x \cos 2x$	For any product of sin and cos with same coefficient of x, use double angle. $\sin 2x\cos 2x \equiv \frac{1}{2}\sin 4x$	$-\frac{1}{8}\cos 4x$
$\sin^5 x \cos x$	Reverse chain rule.	$\frac{1}{6}\sin^6 x$

Year 2 Pure Chapter 12 - Vectors

Describing Co-ordinates

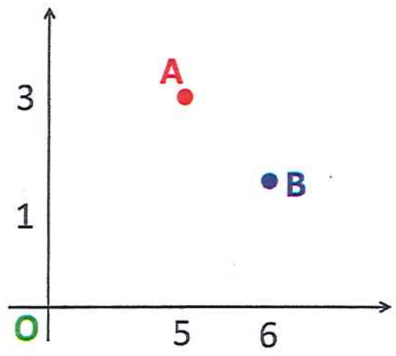


A (3, 2)

or

$0A = 3i + 2j$

Describing Line Segments



AB = A0 + 0B

Magnitude

2D Vector

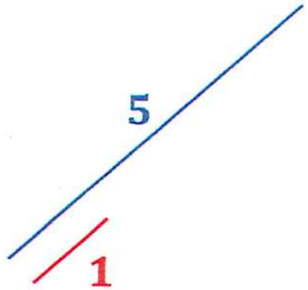
$|a| = \sqrt{i^2 + j^2}$

3D Vector

$|a| = \sqrt{i^2 + j^2 + k^2}$

Unit Vector

A unit vector has a length of 1.



$a = 3i + 4j$

$Unit\ Vector = \frac{1}{5}(3i + 4j)$

$Unit\ Vector = \frac{1}{|a|}(a)$

Vectors and Parallel

Vectors are parallel if one is a multiple of the other.

$a = 3i + 4j + 5k$

$b = 6i + 8j + 10k$
$= 2(3i + 4j + 5k)$

$2a = b$

b is a multiple of a

∴ parallel

Geometric Problems

Show a triangle is a isosceles:
Show only 2 magnitudes are the same.

Show a triangle is a equilateral:
Show only 2 magnitudes are the same.

Show a quadrilateral:
Parallel Sides and Equal Sides

Mechanics

To find the **Resultant Force** add all forces.

$F = ma$

F is a vector

a is a vector

m is a scale